Dynamics of a spheroidal particle in a leaky dielectric medium in an ac electric field

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We study a system comprising a spheroidal particle with permittivity ε_2 and conductivity σ_2 immersed in a host medium with permittivity ε_1 and conductivity σ_1 in the ac external electric field with a strength $\vec{E}_0$ and frequency ω. We determined conditions when orientation of a spheroidal particle with a finite electric conductivity at $t \rightarrow \infty$ coincides with the orientation of the ideal dielectric spheroidal particle, when orientation of a spheroid at $t \rightarrow \infty$ is normal to the orientation of the ideal dielectric spheroidal particle, and when orientation of particle is not affected by the external electric field. We found the direct connection between the final orientation of the particle and the existence of two time intervals, $T_1(\omega)$ and $T_2(\omega)$, such that during time interval T_1 an equilibrium orientation of the particle is the same as the equilibrium orientation of an ideal dielectric particle while during time interval T_2 the direction of the stable equilibrium orientation is normal to the equilibrium orientation of an ideal dielectric particle. The values T_1 and T_2 depend on the frequency of the external field ω and $T = T_1(\omega) + T_2(\omega)$, where $T = 2\pi/\omega$.

I. INTRODUCTION

The dynamics of solid or liquid particles in a host medium under the action of an external electric field is of theoretical and technological interest. Technological applications include manipulation of microparticles in biotechnology and genetic engineering [1], nanotechnology [2,3], and noncontact measurements of physical properties of particles.

The results obtained in numerous theoretical and experimental studies on particle dynamics under the action of the external electric field are summarized in several survey papers and monographs [4–7].

In our previous study [8] we investigated the behavior of various physical parameters, e.g., electric current, dipole moment, and surface charge, in the case when an ellipsoidal particle with a finite electric conductivity σ_2 and permittivity ε_2 is embedded in a host medium with permittivity ε_1 and conductivity σ_1, and the whole system is subjected to the ac electric field with a frequency ω. We found that the stable equilibrium orientation of the particle changes during the period of the external field T. During time interval T_1 an equilibrium orientation of the particle remains the same as the equilibrium orientation of an ideal dielectric particle while during time interval T_2 the direction of the stable equilibrium orientation is normal to the equilibrium orientation of an ideal dielectric particle. The values T_1 and T_2 depend on the frequency of the external electric field ω, and $T = T_1(\omega) + T_2(\omega)$, where $T = 2\pi/\omega$.

In the present study we investigate the dynamics of the particle. In a general case the dynamics of the particle, embedded in a leaky dielectric medium, is quite involved and can become chaotic under the sufficiently large amplitude of the external field as it was showed in Ref. [9]. The behavior of a cylindrical particle under the action of a constant electric

field with the direction normal to the axis of symmetry of the particle was investigated in Ref. [9]. It was demonstrated that even in this relatively simple case, provided the amplitude of the external field is sufficiently large, the system of equations describing the dynamics of the particle is equivalent to a Lorenz's system of equations.

In the present study we consider rotation of the spheroidal particle around one of the axes that is not the axis of symmetry. The vector of the external field, $\vec{E}_0$, is in the plane normal the axis of rotation (see Fig. 1), and we consider a time-dependent external field. Clearly, in the general case, introduction of these additional effects only complicates the behavior of the particle. However, we have found the range of the parameters of the problem whereby the behavior of the particle is determined only by the effect of alternating equilibrium orientations of the particle. In this case the condition $T_2(\omega) > T_1(\omega)$ implies that the final orientation of the particle at $t \rightarrow \infty$ is normal to the orientation of an ideal dielectric particle. In a case when $T_2(\omega) < T_1(\omega)$, the final orientation of the particle coincides with the orientation of the ideal dielectric particle. In this study we determined the depen-
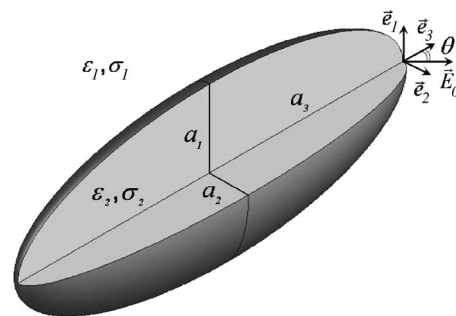

FIG. 1. Ellipsoid with the lengths of semiaxes a_1, a_2, a_3 ($a_1 = a_2 < a_3$–prolate spheroid and $a_1 = a_2 > a_3$–oblate spheroid) and electric permittivity ε_2 and conductivity σ_2 inside a host medium with permittivity ε_1 and conductivity σ_1 in the external electric field $\vec{E}_0$.

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dence of the ratio, $T_2(\omega)/T_1(\omega)$, on the frequency of the external field, ω. We showed that there exist two frequencies, ω_1 and ω_2, whereby $T_2(\omega)=T_1(\omega)$, and, consequently, the electric with these frequencies field does not affect the orientation of the particle. The obtained theoretical results are validated by analyzing the orientation spectra of particles that were overviewed in Ref. [5].

Since the considered problem is of great interest in various biotechnological and nanotechnological applications it was studied in other investigations (see, e.g., Refs. [5,10]). Stability of orientation of the ellipsoidal particle was investigated in Ref. [10] using a formula for energy in the case of the ideal dielectric. Physically such an approach is not consistent since it neglects energy losses caused by the finite electric conductivity. Mathematically, the approach pursued in Ref. [10] is also not well founded since applying Fourier transform to the nonlinear system of equations requires developing a consistent derivation procedure that cannot be reduced to a substitution $\varepsilon \rightarrow \varepsilon_1 + i\varepsilon_2/\omega$.

This investigation is based on the formula for torque, $M(t)$, acting at the particle that was derived in our previous study [8]. This approach allowed us to determine conditions for the monotonic relaxation of the orientation of the particle, and to demonstrate the connection between the final orientation of the particle in a weak electric field and the ratio of time intervals, $T_2(\omega)/T_1(\omega)$. Analysis of the behavior of $T_2(\omega)/T_1(\omega)$ in a wide range of parameters that was performed in the present investigation is of importance in its own right since it allows us to analyze the dynamics of particle when the behavior of particles is more involved than in the case considered in this study.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider an ellipsoidal inclusion with the lengths of the axes a_1, a_2, a_3, permittivity ε_2, and electric conductivity σ_2 that is immersed instantaneously into a host medium with permittivity ε_1 and electric conductivity σ_1 and is subjected to the external electric field $\vec{E}_0$ (see Fig. 1).

The host medium with an embedded particle can be considered as a piece-wise homogeneous medium. Since a charge is localized at the inhomogeneous inclusions, in the case of a piece-wise homogeneous medium it accumulates at the interface boundaries. The density of a surface free charge γ is determined by the following relations:

$$\int \rho_{ex} dV = \int \gamma dS, \quad \text{or} \quad \rho_{ex} = \gamma \delta(u) |\vec{\nabla} u|, \quad (1)$$

where $\delta(u)$ is a Dirac's delta function, $u = F(x,y,z,t)$, and $u=0$ is the equation of the surface.

In a leaky dielectric medium a system of equations that determines the potential component of the electric field, $\vec{E} = -\vec{\nabla}\varphi$, reads [5,6]

$$\vec{\nabla} \cdot \vec{D} = \rho_{ex}, \quad (2)$$

$$\frac{\partial \rho_{ex}}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \quad (3)$$

The electrostatic induction $\vec{D}$ and the electric current density $\vec{j}$ are determined by the following relations:

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E}, \quad \vec{j} = \vec{j}_\sigma + \vec{j}_c, \quad \vec{j}_\sigma = \sigma \vec{E}, \quad \vec{E} = -\vec{\nabla}\varphi, \quad (4)$$

where $\vec{j}_c$ is a convective electric current caused by a macroscopic motion of the charged particles.

In the case of a rotating particle which is considered in this study,

$$\vec{j}_c = \vec{v}_s \gamma \delta(u) |\vec{\nabla} u|, \quad \vec{v}_s = \vec{\Omega} \times \vec{r}_s, \quad (5)$$

where $\vec{\Omega}$ is the angular velocity of the particle and $\vec{r}_s$ is the radius vector at the particle's surface.

Equations (1), (4), and (5) yield the following boundary conditions for Eqs. (2), (3) on the interface boundary:

$$[\vec{N} \cdot \vec{D}] = \gamma, \quad [\vec{N} \cdot \vec{j}_\sigma] = -\frac{\partial \gamma}{\partial t} - \vec{v}_s \cdot (\vec{\nabla} - \vec{N} \cdot \vec{\nabla})\gamma. \quad (6)$$

Here $[A] = A_+ - A_-$, A_+, and A_- are values of the function A at the external and the internal surfaces, respectively, and $\vec{N}$ is an external unit normal vector to the particle's surface.

In order to solve the above problem it is convenient to switch to a frame associated with the particle whereby the axes of coordinates rotate together with the ellipsoid and are directed along the principal axes of the ellipsoid. In this frame $\vec{v}_s = 0$, and the equations of the surface of the ellipsoid $u'(x', y', z', t)$ and the components of the electric field read

$$u' = \sum_{i=1}^{3} \frac{x_i'^2}{a_i^2} - 1, \quad \vec{E} = \sum_{i=1}^{3} E_i' \vec{e}_i', \quad \vec{e}_i' = \vec{\nabla} x_i', \quad (7)$$

where $\vec{e}_i'(t)$ are unit vectors directed along the axes of the ellipsoid. The components of electric field in the rotating frame, E_i', are determined through the components of electric field in the laboratory frame, E_k, using the transformation of the rotation, and for the case considered in this study these expressions are written below [see Eqs. (20)]. Instead of the boundary conditions (6), in the rotating frame we obtain the following boundary conditions:

$$[\vec{N}' \cdot \vec{D}] = \gamma', \quad [\vec{N}' \cdot \vec{j}_\sigma] = -\frac{\partial \gamma'}{\partial t}, \quad (8)$$

where $\vec{N}'$ is an external unit normal vector to the particle's surface in the system of coordinates x', y', z', and γ' is a distribution of the surface charge in the rotating frame.

A system of equations (2), (3), relations (4) with $\vec{\gamma}_c = \vec{0}$, and boundary conditions (8) provide the mathematical formulation of the problem that is considered in this study. After finding the solution of this boundary value problem the solution in the laboratory frame can be obtained by changing variables that are connected through the formulas of the transformation of rotation. Clearly while in the rotating frame the solution satisfies the boundary conditions (8), in the laboratory frame the solution satisfies the boundary con-

ditions (5). In order to simplify the notations hereafter we omit primes near the variables in the rotating frame of reference.

The scalar potential in the ellipsoid and in the host medium is governed by Laplace equation $\vec{\nabla}^2 \varphi = 0$, and it is convenient to represent a scalar potential as

$$\varphi = \varphi_\varepsilon + \varphi_\sigma. \tag{9}$$

Here φ_ε is a potential caused by a local polarization occurring on the microscopic time scale, and it is equal to the total potential in the case of the ideal dielectric:

$$\varphi_\varepsilon = -\sum_{i=1}^{3} \frac{E_{0i}x_i}{1+f_{i\varepsilon}} \left\{ \theta(-\xi) + \theta(\xi)\left[1 + f_{i\varepsilon}\left(1 - \frac{I_i(\xi)}{I_i(0)} \right) \right] \right\}, \tag{10}$$

where

$$I_i(\xi) = \int_\xi^\infty \frac{ds}{(s+a)R(s)}, \quad R(s)$$

$$= \sqrt{(s+a_1^2)(s+a_2^2)(s+a_3^2)}, \quad f_{i\varepsilon} = \kappa_\varepsilon n_i,$$

$$\kappa_\varepsilon = \frac{\varepsilon_2}{\varepsilon_1} - 1,$$

$$n_i = R(0)I_i(0)/2$$

is a depolarization coefficient of a spheroid,

$$0 < n_i < 1, \quad \sum_{i=1}^{3} n_i = 1, \quad \theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases},$$

ξ is the ellipsoidal coordinate determined through x_1, x_2, x_3 by formulas presented in Ref. [11] (Chap. 1, Sec. IV), and ξ is chosen such that $\xi = 0$ corresponds to a surface of the spheroid $u = 0$.

The additional potential φ_σ arises due to the finite conductivity, and it is caused by the accumulation of the electric charge at the surface of the ellipsoidal particle. The expression for φ_σ reads

$$\varphi_\sigma = -\sum_{i=1}^{3} \frac{x_i(f_{i\varepsilon} - f_{i\sigma})\Pi_i(t)}{(1+f_{i\sigma})(1+f_{i\varepsilon})} \left[\theta(-\xi) + \frac{I_i(\xi)}{I_i(0)}\theta(\xi) \right]. \tag{11}$$

Here $f_{i\sigma} = \kappa_\sigma n_i$, $\kappa_\sigma = \frac{\sigma_2}{\sigma_1} - 1$, and the function $\Pi_i(t)$ is determined through the components of the electric field along the principal axes of the ellipsoid, E_{0i}:

$$\frac{d\Pi_i(t)}{dt} + \frac{\Pi_i(t)}{\tau_i} = \frac{E_{0i}}{\tau_i}, \tag{12}$$

where the relaxation times of the surface charges

$$\tau_i = \tau_0 \frac{1+f_{i\varepsilon}}{1+f_{i\sigma}}, \quad \tau_0 = \frac{\varepsilon_0\varepsilon_1}{\sigma_1}.$$

As it has been noted above, formulas (6)–(11) and the equations below are written in a frame associated with the rotating ellipsoid. Therefore x_i in Eq. (11) are coordinates in the

system where the axes are directed along the principal axes of the ellipsoid.

According to Eqs. (4) the scalar potential φ determines all electrodynamic parameters of the problem.

In the analysis of the dynamics of the rotating ellipsoid we use the concept of the dipole moment whereby the expression for torque acting at the ellipsoid reads [5]

$$\vec{M} = \varepsilon_1 \vec{P} \times \vec{E}_0, \tag{13}$$

where $\vec{P}$ is a dipole moment of the ellipsoid. Using Eqs. (1) the expression for the dipole moment $\vec{P}$ can be written as follows:

$$\vec{P} = \int \gamma_T \vec{r} dS, \tag{14}$$

where integration is performed over the surface of the ellipsoid and γ_T is surface density of the total charge on the surface:

$$\gamma_T = \varepsilon_0 [\vec{E}] \cdot \vec{N}. \tag{15}$$

Similarly to the scalar potential φ, an expression for the dipole moment and other physical parameters can be written as follows:

$$\vec{P} = \vec{P}_\varepsilon + \vec{P}_\sigma. \tag{16}$$

Using Eqs. (10), (11), (14), and (16) we arrive at the following expressions for $\vec{P}_\varepsilon$ and $\vec{P}_\sigma$:

$$\vec{P}_\varepsilon = \varepsilon_0 V \sum_{i=1}^{3} \frac{E_{0i}\vec{e}_i\kappa_\varepsilon}{1+f_{i\varepsilon}}, \quad \vec{P}_\sigma = \varepsilon_0 V \sum_{i=1}^{3} \frac{\Pi_i\vec{e}_i(\kappa_\sigma - \kappa_\varepsilon)}{(1+f_{i\varepsilon})(1+f_{i\sigma})}. \tag{17}$$

Now employing Eq. (13) one can derive a general expression for the torque acting at the particle, $\vec{M}$. This formula was derived in Ref. [8], and in the present study we consider only a special case. Assume that a particle is a spheroid with the axis of symmetry directed along the unit vector $\vec{e}_3$. The axis of rotation coincides with one of the principal axes of the spheroid and is directed along the unit vector $\vec{e}_2$ ($\vec{e}_3 = \vec{e}_1 \times \vec{e}_2$).

Without the loss of generality it can be assumed that the electric field $\vec{E}_0$ is normal to this axis. Equations (13) and (16) yield the following expressions for the total torque acting at the particle, $\vec{M} = M\vec{e}_2$:

$$M = M_\varepsilon + M_\sigma, \tag{18}$$

$$M_\varepsilon = \frac{\varepsilon_0\varepsilon_1 V E_{03}E_{01}\kappa_\varepsilon^2(1-3n)}{(1+f_{3\varepsilon})(1+f_{1\varepsilon})},$$

$$M_\sigma = \varepsilon_0\varepsilon_1 V(\kappa_\sigma - \kappa_\varepsilon)\left(\frac{E_{01}\Pi_3}{(1+f_{3\sigma})(1+f_{3\varepsilon})} \right.$$

$$\left. - \frac{E_{03}\Pi_1}{(1+f_{1\sigma})(1+f_{1\varepsilon})} \right). \tag{19}$$

Let us measure a rotation angle θ from the position when direction of $\vec{e}_3$ coincides with the direction of the external electric field $\vec{E}_0$, and as the positive direction we select the direction of the counterclockwise rotation with respect to the axis $\vec{e}_2$. Then

$$\vec{E}_0(t) = E_0(t)\cos[\theta(t)]\vec{e}_3 - E_0(t)\sin[\theta(t)]\vec{e}_1. \qquad (20)$$

The case when the external field is given by Eq. (20) formulas for functions $\Pi_1(t)$ and $\Pi_3(t)$ read

$$\Pi_1(t) = -\frac{1}{\tau_1}\int_0^t \exp\left(-\frac{\tau}{\tau_1}\right)E_0(t-\tau)\sin[\theta(t-\tau)]d\tau,$$

$$(21)$$

$$\Pi_3(t) = \frac{1}{\tau_3}\int_0^t \exp\left(-\frac{\tau}{\tau_3}\right)E_0(t-\tau)\cos[\theta(t-\tau)]d\tau. \qquad (22)$$

In the following we will focus on the investigation of the behavior of noninertial particles.

III. DYNAMICS OF THE SPHEROIDAL PARTICLE IN A WEAK AC FIELD

Relaxation equation for the dynamics of noninertial particle reads

$$\eta_s\dot{\theta} = M(t), \qquad (23)$$

where $\eta_s\dot{\theta}$ is a torque acting at the particle due to the viscosity of a host fluid. $M(t)$ is a torque acting at the particle due to the external electric field.

Inspection of Eqs. (18)–(23) shows that particle dynamics is quite involved even in the case of noninertial particles. The situation is considerably simplified when the following conditions are satisfied:

$$\tau_\eta \gg \frac{1}{\omega}, \quad \tau_\eta \gg \tau_1, \quad \tau_\eta \gg \tau_3, \quad \tau_\eta = \frac{2\eta_S}{\varepsilon_0\varepsilon_1 V\tilde{E}_0^2}, \quad (24)$$

where ω is a frequency of the external electric field, τ_η is the effective relaxation time of the orientation of the particle [see Eq. (34) below], V is a volume of a particle, $\tilde{E}_0$ is the amplitude of the external field, and it is assumed that the time dependence of the external field is given by $E_0(t) = \tilde{E}_0\cos(\omega t)$.

In the following we have shown that under these conditions time variation of the angle, $\theta(t)$, occurs much slower in comparison with the period of the external field. Consequently, $\theta(t-\tau)$ in Eqs. (21) and (22) can be replaced by $\theta(t)$, and functions $\Pi_1(t)$ and $\Pi_3(t)$ can be easily calculated. For simplicity we assume that $t \gg \tau_1, \tau_3$. In this limit

$$\Pi_1(t) = -\tilde{\Pi}_i(t)\sin[\theta(t)], \quad \Pi_3(t) = \tilde{\Pi}_3(t)\cos[\theta(t)],$$

$$\tilde{\Pi}_i(t) = \frac{\cos(\omega t) + \omega\tau_i\sin(\omega t)}{1 + \omega^2\tau_i^2}. \qquad (25)$$

Then using Eqs. (18) and (19) the equation for $M(t)$ can be written as follows:

$$M = M_0\frac{3n-1}{2}[M_c + 2M_s\tan(\omega t)]\cos^2(\omega t), \qquad (26)$$

where

$$M_0 = \frac{\varepsilon_0\varepsilon_1 V\tilde{E}_0^2\sin(2\theta)}{2}, \quad M_c = \frac{M_{c0}}{L} + \frac{M_{c2}\omega^2\tau_0^2}{L} + \frac{M_{c4}\omega^4\tau_0^4}{L},$$

$$M_s = \frac{b_0\omega\tau_0}{L}(b_1\kappa_\sigma + b_2\kappa_\varepsilon\omega^2\tau_0^2)(\kappa_\sigma - \kappa_\varepsilon). \qquad (27)$$

The parameter L in Eqs. (27) is determined by the following formula:

$$L = (1 + f_{3\sigma})(1 + f_{1\sigma})(1 + f_{3\varepsilon})(1 + f_{1\varepsilon})(1 + \omega^2\tau_1^2)(1 + \omega^2\tau_3^2), \qquad (28)$$

and expressions for M_{c0}, M_{c2}, M_{c4} read

$$M_{c0} = \kappa_\sigma^2(1 + f_{3\varepsilon})(1 + f_{1\varepsilon}), \qquad (29)$$

$$M_{c2} = \frac{(\kappa_\sigma - \kappa_\varepsilon)(3\kappa_\varepsilon - \kappa_\sigma + 3d_1\kappa_\varepsilon^2 + d_2\kappa_\varepsilon^3)}{(1 + f_{3\sigma})(1 + f_{1\sigma})}$$
$$+ \kappa_\varepsilon^2\frac{(1 + f_{3\varepsilon})^2(1 + f_{1\sigma})^2 + (1 + f_{3\sigma})^2(1 + f_{1\varepsilon})^2}{(1 + f_{3\sigma})(1 + f_{1\sigma})}, \qquad (30)$$

$$M_{c4} = \kappa_\varepsilon^2\frac{(1 + f_{3\varepsilon})^2(1 + f_{1\varepsilon})^2}{(1 + f_{3\sigma})(1 + f_{1\sigma})} > 0, \qquad (31)$$

where coefficients b_0, b_1, b_2, d_1, d_2 are given by the following formulas:

$$b_0 = \frac{1}{2}\frac{(1 + f_{3\varepsilon})(1 + f_{1\varepsilon})}{(1 + f_{3\sigma})(1 + f_{1\sigma})},$$

$$b_1 = 2 + \kappa_\sigma\frac{1+n}{2},$$

$$b_2 = 2 + \kappa_\varepsilon\frac{1+n}{2}, \qquad (32)$$

$$d_1 = \frac{1+n}{2} + \kappa_\sigma\frac{n(1-n)}{2},$$

$$d_2 = \frac{(1-n)^2}{4} + n^2 + \frac{n(1-n)}{2}\left(1 + \kappa_\sigma\frac{1+n}{2}\right). \qquad (33)$$

Equations (26)–(33) describe the behavior of the torque, $M(t)$, at any time provided that condition (24) is satisfied and, consequently, $\Pi_i(t)$ are determined by Eq. (25). The expression for M_c in this study is identical to the formula that was derived in Ref. [8] but it is written in a more convenient for further analysis form.

Since the angle $\theta(t)$ in the approximation given by Eq. (24) appears only in the formula for M_0 [see the first formula in Eqs. (27)], Eq. (23) can be easily solved. Then we obtain

$$|\tan(\theta)| = |\tan(\theta_0)|\exp\left[\frac{(3n-1)}{2}\left(M_c\frac{t}{\tau_\eta}\right.\right.$$
$$\left.\left.+\frac{\sin(2\omega t - \varphi)}{\omega\tau_\eta}\,\mathrm{sgn}(3n-1)\sqrt{\frac{M_c^2}{4}+M_s^2}\right)\right],$$
(34)

where $\tan(\varphi) = 2M_s/M_c$ and θ_0 is the initial angle at $t=0$.

Equation (34) can be also rewritten as follows:

$$|\tan(\theta)| = |\tan(\theta_0)|\exp\left(\frac{3n-1}{2}M_c\frac{t}{\tau_\eta}\right)B(t),$$
(35)

where $B(t) = \sum\limits_{k=-\infty}^{\infty} I_k\left(\frac{|3n-1|}{2\omega\tau_\eta}\sqrt{\frac{M_c^2}{4}+M_s^2}\right)\exp\left[ik\left(2\omega t - \varphi - \frac{\pi}{2}\right)\right]$ and $I_k(z)$ are modified Bessel functions [12]. For slow variations of $\sin[\theta(t)]$ and $\cos[\theta(t)]$ that were assumed in derivation of Eq. (25), the argument must be small, $z \ll 1$. The latter condition yields an additional condition to Eqs. (24):

$$\sqrt{\frac{M_c^2}{4}+M_s^2} \sim 1.$$
(36)

Under conditions (24), (36) the amplitude of the oscillations with the frequency $2k\omega$ is of the order of $(1/\omega\tau_\eta)^k$, and then it can be assumed that $B(t)=1$.

Therefore when conditions (24), (36) are satisfied and $t \to \infty$, the particle monotonically approaches the state with $\theta=0$ or $\theta=\pi/2$. In the case of a prolate spheroid, $n<1/3$, $\theta \to 0$ when $M_c>0$ and $\theta \to \pi/2$ when $M_c<0$. Similar analysis performed for the case $n>1/3$ shows that a condition $M_c<0$ corresponds to the orientation perpendicular to the orientation in the case of the ideal dielectric. If $M_c=0$, the external electric field does not affect the orientation of the particle.

The physical reason for this behavior is directly associated with the existence of time intervals $T_1(\omega)$ and $T_2(\omega)$ such that stable equilibrium orientation of the particle during time interval $T_1(\omega)$ coincides with that in the case of the ideal dielectric particle while during time interval $T_2(\omega)$ stable equilibrium orientation of the particle is normal to the equilibrium orientation of an ideal dielectric particle. The existence of these time intervals can be deduced from Eq. (26). Indeed, Eq. (26) implies that if the following condition is satisfied:

$$M_c + 2M_s\tan(\omega t) > 0,$$
(37)

then the equilibrium orientation of the spheroid is the same as in the case of the ideal dielectric particle while in the opposite case the equilibrium orientation of the spheroid is directed perpendicular to the orientation of the ideal dielectric particle.

Using the inequality (37) we can determine the time interval $T_1(\omega)$ during which inequality (37) is satisfied, and time interval $T_2(\omega)$ when the inequality is violated:

$$\frac{T_1}{T} = \frac{1}{2} + \frac{M_c}{\pi|M_c|}\tan^{-1}\left|\frac{M_c}{2M_s}\right|,$$
$$\frac{T_2}{T} = \frac{1}{2} - \frac{M_c}{\pi|M_c|}\tan^{-1}\left|\frac{M_c}{2M_s}\right|.$$
(38)

Equations (38) imply that when $M_c<0$, $T_2>T_1$ and spheroid is oriented perpendicular to the orientation of the ideal dielectric particle. Equations (27) yield the following formula for M_c:

$$M_c = \frac{M_{c4}}{L}(\nu^2 - \nu_1^2)(\nu - \nu_2^2),$$
(39)

where $\nu = \omega\tau_0$,

$$\nu_1^2 = -\frac{M_{c2}}{2M_{c4}} - \sqrt{\frac{1}{4}\left(\frac{M_{c2}}{M_{c4}}\right)^2 - \frac{M_{c0}}{M_{c4}}},$$
(40)

and

$$\nu_2^2 = -\frac{M_{c2}}{2M_{c4}} + \sqrt{\frac{1}{4}\left(\frac{M_{c2}}{M_{c4}}\right)^2 - \frac{M_{c0}}{M_{c4}}}.$$
(41)

Equations (29), (31) imply that $M_{c0}>0$ and $M_{c4}>0$. Therefore when the following conditions are met:

$$M_{c2}^2 > M_{c0} \cdot M_{c4}, \quad M_{c2} < 0,$$
(42)

then $\nu_1^2>0$, $\nu_2^2>0$, and $T_2(\omega)>T_1(\omega)$ in the frequency range $\nu_1 < \nu < \nu_2$. In this frequency range and when conditions (42) are met a particle is oriented normally to the orientation of the ideal dielectric particle. If conditions (42) are violated, the frequency range where $T_2(\omega)>T_1(\omega)$ does not exist, and in the whole frequency range the final orientation of the particle is the same as in the case of the ideal dielectric particle.

To the best of our knowledge the direct investigations of the dynamics of inversion of the orientation of the spheroidal particle in an ac field caused by a finite electric conductivity were not conducted as yet. However, the theoretical results obtained in the present study can be compared with the experimental data qualitatively by analyzing orientation spectra described in Refs. [5,13]. Monograph [5] (p. 125, Fig. 5.5) discusses orientation spectra of small titanium dioxide particles suspended in isopropanol. Inspection of Fig. 5.5 shows that for small and large frequencies the particles are oriented as in the case of the ideal dielectric particles while for the intermediate frequencies particles are oriented in the normal direction in compliance with the condition $\nu_1 < \nu < \nu_2$, where $M_c<0$. Moreover, these regions are separated by a narrow interval of frequencies with indefinite orientation of the particles. It is conceivable to suggest that the latter observation corresponds to the condition $M_c=0$ or $T_1(\omega)=T_2(\omega)$. Other experimental results (see, e.g., Refs. [5,13]) are ambiguous although some regions of the orientation spectra in these study comply with the developed theory. Exact quantitative comparison of the derived theoretical results with the experimental data cannot be performed because of the following two reasons. The first reason is that the accuracy of measurements of electric conductivity of microparticles, σ_2, is not sufficiently high. Thus, e.g., in Refs. [5,13] electric conductivity of particles was measured in S/m while electric con-
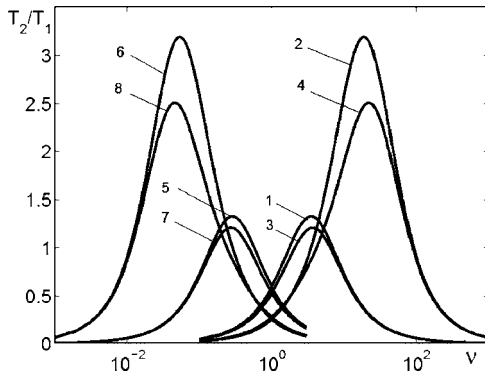
FIG. 2. Dependence of the ratio T_2/T_2 vs frequency of the external electric field ν. $1-\kappa_\varepsilon=0.1$, $\kappa_\sigma=1$, $n=0.1$; $2-\kappa_\varepsilon=0.1$, $\kappa_\sigma=10$, $n=0.1$; $3-\kappa_\varepsilon=0.1$, $\kappa_\sigma=1$, $n=0.9$; $4-\kappa_\varepsilon=0.1$, $\kappa_\sigma=10$, $n=0.9$; $5-\kappa_\varepsilon=1$, $\kappa_\sigma=0.1$, $n=0.1$; $6-\kappa_\varepsilon=10$, $\kappa_\sigma=0.1$, $n=0.1$; $7-\kappa_\varepsilon=1$, $\kappa_\sigma=0.1$, $n=0.9$; $8-\kappa_\varepsilon=10$, $\kappa_\sigma=0.1$, $n=0.9$.

ductivity of the solution was measured in μS/m. The second reason is that the shape of the particle employed in Refs. [5,13] is close to ellipsoidal while in this study we investigated the case of a spheroidal particle. It must be noted that the analysis presented below shows that geometry plays only a minor role. Thus, e.g., the ratio of the temporal intervals, $T_1(\omega)/T_2(\omega)$, as a function of frequency, only weakly depends upon a polarization factor, n. This means that for the same magnitudes of the parameters κ_ε and κ_σ, a particle changes its transversal orientation to the opposite one in approximately the same range of frequencies independent on the geometry.

In Fig. 2 we showed the dependence of $T_1(\omega)/T_2(\omega)$ vs $\nu=\omega\tau_0$ for several values of parameters κ_ε and κ_σ. Inspection of this figure shows that the inequality $T_2(\nu)>T_1(\nu)$ is attained for $\kappa_\varepsilon>\kappa_\sigma$ as well as for $\kappa_\varepsilon<\kappa_\sigma$. Figure 3 shows the dependence of $T_1(\omega)/T_2(\omega)$ vs $\nu=\omega\tau_0$ for other values of the parameters κ_ε and κ_σ whereby $T_2(\nu)<T_1(\nu)$ in the whole frequency range.

The derived equations yield the asymptotic formulas for the ratio $T_1(\omega)/T_2(\omega)$ for small, $\omega\tau_0\to 0$, and large, $\omega\tau_0\to\infty$, frequencies ω. At small frequencies when $\kappa_\sigma\gtrsim 1$, $\omega\tau_0\ll 1$, or when $\kappa_\sigma\ll 1$, $\omega\tau_0\ll\kappa_\sigma$,
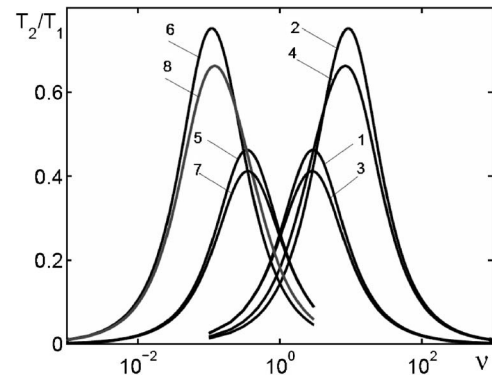

FIG. 3. Dependence of the ratio T_2/T_1 vs frequency of the external electric field ν. $1-\kappa_\varepsilon=1$, $\kappa_\sigma=5$, $n=0.1$; $2-\kappa_\varepsilon=1$, $\kappa_\sigma=20$, $n=0.1$; $3-\kappa_\varepsilon=1$, $\kappa_\sigma=5$, $n=0.9$; $4-\kappa_\varepsilon=1$, $\kappa_\sigma=20$, $n=0.9$; $5-\kappa_\varepsilon=5$, $\kappa_\sigma=1$, $n=0.1$; $6-\kappa_\varepsilon=20$, $\kappa_\sigma=1$, $n=0.1$; $7-\kappa_\varepsilon=5$, $\kappa_\sigma=1$, $n=0.9$; $8-\kappa_\varepsilon=20$, $\kappa_\sigma=1$, $n=0.9$.

$$\frac{T_2}{T_1} = \frac{1}{\pi}\frac{|\kappa_\varepsilon - \kappa_\sigma|}{\kappa_\sigma}\frac{\omega\tau_0\left(2 + \kappa_\sigma\dfrac{n+1}{2}\right)}{(1+f_{1\sigma})(1+f_{3\sigma})}. \tag{43}$$

At large frequencies when $\omega\tau_0\to\infty$

$$\frac{T_2}{T_1} = \frac{4}{\pi}\frac{b_0 b_2|\kappa_\varepsilon||\kappa_\sigma - \kappa_\varepsilon|}{M_{c4}\omega\tau_0}. \tag{44}$$

IV. CONCLUSIONS

We investigated dynamics of a dielectric particle with a finite electric conductivity immersed in a leaky dielectric medium subjected to a weak ac external electric field when the behavior of the particle reduces to a monotone relaxation. We derived conditions for a monotone relaxation, Eq. (24), and additional condition (36). Since the effective relaxation time τ_η depends on the amplitude of the electric field [see Eqs. (24)] the increase of the amplitude of the external electric field is accompanied by a reduction of $\tau_\eta \sim 1/E^2$. The latter, according to Eq. (34), implies excitation of the high frequency oscillations and rapid variations of $\sin[\theta(t)]$ and $\cos[\theta(t)]$ in this range of parameters. Clearly under these circumstances $\theta(t-\tau)$ cannot be replaced by $\theta(t)$, and the analysis of particle dynamics under these conditions is quite involved and warrants a separate study.

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